Bi-Hamiltonian Structure of Gradient Systems in Three Dimensions and Geometry of Potential Surfaces

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Abstract:
Working bi-Hamiltonian structure and Jacobi identity in Frenet-Serret frame associated to a dynamical system, we proved that all dynamical systems in three dimensions possess two compatible Poisson structures. We investigate relations between geometry of surfaces defined by potential function of a gradient system and its bi-Hamiltonian structure. We show that it is possible to find Hamiltonian functions whose gradient flows have geodesic curvature zero on potential surfaces. Using this, we conclude that Hamiltonian functions are determined by distance functions on potential surfaces. We apply this technique to find conserved quantities of three dimensional gradient systems including the Aristotelian model of the three-body motion.

1 Introduction

A Poisson structure on a manifold is defined by a skew symmetric contravariant bilinear form subjected to the Jacobi identity expressed as the vanishing of the Schouten bracket of Poisson tensor with itself [1]-[4]. This structure having no non-degeneracy requirement becomes the basic underlying geometry to study non-canonical Hamilton's equations on odd dimensional manifolds as well as the Hamiltonian structures of nonlinear evolution equations [3]-[5].

The first interesting case of a completely degenerate finite dimensional Hamiltonian structure occurs in three dimensions. Many works have been devoted to the study of three dimensional dynamical systems with primary concern on quantization, construction of conserved quantities, Hamiltonian structures, integrability problems and their numerical integration using techniques from various areas such as Poisson geometry, differential equations, Frobenius integrability theorem and theory of foliations [6]-[26].

In [27], we reduced the problem of constructing Hamiltonian structures in three dimensions to the solutions of a Riccati equation in moving coordinates of

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Frenet-Serret frame. All known examples of dynamical systems having two compatible and explicit Hamiltonian structures are exhausted by constant solution. We concluded that in three dimensions vector fields which are not eigenvectors of the curl operator are at least locally bi-Hamiltonian. This structure manifests itself in the well-known form of Frenet-Serret triad $\mathbf{t} = \mathbf{n} \times \mathbf{b}$ where $\mathbf{t}$ is the unit tangent vector associated with the given dynamical system and, the normal vectors $\mathbf{n}$ and $\mathbf{b}$ are related to conserved covariants. From the expression of gradient operator in Frenet-Serret frame (c.f. Eq.(3)) we observe that unit tangent vector arises from the Cartesian gradient of a function of arclenght variable.

In this work, we shall continue to investigate the local structure of bi-Hamiltonian systems in three dimensions assuming that they are described by gradient vector fields. We shall relate ingredients of bi-Hamiltonian structure to geometry of surfaces described by the potential function F. In particular, we shall prove that there are two Hamiltonian functions related with geodesic coordinates on the potential surfaces.

1.1 Content of the work

In the next section, we shall first develop differential calculus in Frenet-Serret frame that will be used throughout the paper. Then, we shall survey on conditions for construction of Frenet-Serret frame for a given vector field in $\mathbb{R}^3$ to ensure its existence. This will extend our previous result in [27] that excludes eigenvectors of curl operator.

In Section 3, we shall summarize ingredients of bi-Hamiltonian systems in three dimensions. In particular, we shall identify a Poisson bi-vector with a locally integrable (in the sense of Frobenius) vector field in three dimensions and, will work with the latter. We shall first reduce the Jacobi identity into Riccati equation and then, assuming we have independent solutions, we shall exhibit relations between conserved Hamiltonians and Poisson vectors.

In Section 4, we shall start with geometric characterization of potential surfaces in the normal coordinates of the Frenet-Serret frame. By considering gradient flows of Hamiltonian functions on potential surfaces, we shall prove that it is possible to find Hamiltonian functions whose gradient flows on the potential surface have geodesic curvature zero. Using this result, we shall conclude that Hamiltonian functions are determined by distance functions on potential surfaces. Finally, we shall prove that Hamiltonian functions of the gradient system are related with geodesic coordinates of the potential surfaces.

In Sections 5, we shall present examples of gradient dynamical systems which are bi-Hamiltonian and, work out in details the geometry of potential surface.

2 Frenet-Serret Frame

Let $\mathbf{t}(x, y, z)$ be a given unit vector field in $\mathbb{R}^3$ endowed with Cartesian coordinates $\mathbf{x} = (x, y, z)$. We may assume $\mathbf{t}$ to be a unit tangent vector to a curve $t \to \mathbf{x}(t)$ in $\mathbb{R}^3$. This may be the solution of dynamical system in Eq.(18). In

this case, we can choose $\mathbf{t} = \mathbf{v}/||\mathbf{v}||$. Locally, one can always lift $\mathbf{t}$ to an orthonormal frame in $\mathbb{R}^3$ in infinitely many ways. Let $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be such an arbitrary orthonormal frame satisfying

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}.$$

We introduce the directional derivatives along the triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ as

$$\partial_s = \mathbf{t} \cdot \nabla \qquad \partial_n = \mathbf{n} \cdot \nabla \qquad \partial_b = \mathbf{b} \cdot \nabla \tag{1}$$

so that the variables (s, n, b) are the coordinates associated with the Frenet-Serret frame. Assuming the Cartesian coordinates are functions $\mathbf{x} = \mathbf{x}(s, n.b)$ of Frenet-Serret coordinates we find, using Eq.(1) the Jacobian matrix

$$\frac{\partial(x,y,z)}{\partial(s,n,b)} = \begin{vmatrix} \partial_s x & \partial_n x & \partial_b x \\ \partial_s y & \partial_n y & \partial_b y \\ \partial_s z & \partial_n z & \partial_b z \end{vmatrix} = \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{vmatrix}$$

whose determinant

$$\det\left(\frac{\partial(x,y,z)}{\partial(s,n,b)}\right) = \mathbf{t}\cdot(\mathbf{n} \times \mathbf{b}) = 1$$

is non-zero and hence the inverse transformation

$$s = s(\mathbf{x}), \quad n = n(\mathbf{x}), \quad b = b(\mathbf{x}) \tag{2}$$

exists locally, that is, in a sufficiently small neighborhood of a given point $\mathbf{x}_0 \in \mathbb{R}^3$. These functions may be obtained by integrating the quantities

$$ds = \mathbf{t} \cdot d\mathbf{x}, \qquad dn = \mathbf{n} \cdot d\mathbf{x}, \qquad db = \mathbf{b} \cdot d\mathbf{x}$$

the last two of which implies n =constant and b =constant when restricted to the curve $\mathbf{x}(t)$.

2.1 Differential calculus

By inverting equations (1) we get the expression

$$\nabla = \mathbf{t}\partial_s + \mathbf{n}\partial_n + \mathbf{b}\partial_b \tag{3}$$

for the Cartesian gradient in Frenet-Serret frame. For future reference, we define the helicities [27]

$$\mathcal{H}_t = \mathbf{t} \cdot \nabla \times \mathbf{t}, \quad \mathcal{H}_n = \mathbf{n} \cdot \nabla \times \mathbf{n}, \quad \mathcal{H}_b = \mathbf{b} \cdot \nabla \times \mathbf{b} \tag{4}$$

and the cross-helicities

$$\begin{aligned} \mathcal{H}_{tn} &= \mathbf{t} \cdot \nabla \times \mathbf{n}, \quad \mathcal{H}_{nt} = \mathbf{n} \cdot \nabla \times \mathbf{t}, \quad \mathcal{H}_{nb} = \mathbf{n} \cdot \nabla \times \mathbf{b} \\ \mathcal{H}_{tb} &= \mathbf{t} \cdot \nabla \times \mathbf{b}, \quad \mathcal{H}_{bt} = \mathbf{b} \cdot \nabla \times \mathbf{t}, \quad \mathcal{H}_{bn} = \mathbf{b} \cdot \nabla \times \mathbf{n}. \end{aligned} \tag{5}$$

which measure holonomicity of Frenet-Serret triad. From the coefficients of the basis vectors in expansions of curls into $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ we obtain

$$
\begin{aligned}
\nabla \times \mathbf{t} &= \mathcal{H}_t \mathbf{t} + \mathcal{H}_{nt} \mathbf{n} + \mathcal{H}_{bt} \mathbf{b} \\
\nabla \times \mathbf{n} &= \mathcal{H}_{tn} \mathbf{t} + \mathcal{H}_n \mathbf{n} + \mathcal{H}_{bn} \mathbf{b} \\
\nabla \times \mathbf{b} &= \mathcal{H}_{tb} \mathbf{t} + \mathcal{H}_{nb} \mathbf{n} + \mathcal{H}_b \mathbf{b}
\end{aligned}
\tag{6}
$$

and from the orthonormality of basis vectors we have the divergences

$$
\begin{aligned}
\nabla \cdot \mathbf{t} &= \mathcal{H}_{bn} - \mathcal{H}_{nb} \\
\nabla \cdot \mathbf{n} &= \mathcal{H}_{tb} - \mathcal{H}_{bt} \\
\nabla \cdot \mathbf{b} &= \mathcal{H}_{nt} - \mathcal{H}_{tn}.
\end{aligned}
\tag{7}
$$

Using the vector identity

$$
\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \tag{8}
$$

we compute the derivative of $\mathbf{t}$ in direction of $\mathbf{t}$ to be

$$
\partial_s \mathbf{t} = (\mathbf{t} \cdot \boldsymbol{\nabla})\mathbf{t} = -\mathbf{t} \times (\nabla \times \mathbf{t}) = -(\mathbf{n} \times \mathbf{b}) \times (\nabla \times \mathbf{t}) = \mathbf{n}\mathcal{H}_{bt} - \mathbf{b}\mathcal{H}_{nt}.
$$

Repeating for the other directions we have the derivatives

$$
\begin{aligned}
\partial_s \mathbf{t} &= \mathbf{n}\mathcal{H}_{bt} - \mathbf{b}\mathcal{H}_{nt} \\
\partial_n \mathbf{n} &= \mathbf{b}\mathcal{H}_{tn} - \mathbf{t}\mathcal{H}_{bn} \\
\partial_b \mathbf{b} &= \mathbf{t}\mathcal{H}_{nb} - \mathbf{n}\mathcal{H}_{tb}
\end{aligned}
\tag{9}
$$

of basis vectors along their respective directions. To find other derivatives, add the identity

$$
\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}
$$

to Eq.(8) to obtain

$$
\begin{aligned}
2(\mathbf{v} \cdot \nabla)\mathbf{u} &= \nabla(\mathbf{u} \cdot \mathbf{v}) + \nabla \times (\mathbf{u} \times \mathbf{v}) - (\nabla \cdot \mathbf{v})\mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{v} \\
&\quad -\mathbf{u} \times (\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{u})
\end{aligned}
$$

and let $\mathbf{u} = \mathbf{t}$, $\mathbf{v} = \mathbf{n}$ to have

$$
\begin{aligned}
2(\mathbf{n} \cdot \nabla)\mathbf{t} &= 2\partial_n \mathbf{t} \\
&= \nabla \times \mathbf{b} - (\nabla \cdot \mathbf{n})\mathbf{t} + (\nabla \cdot \mathbf{t})\mathbf{n} - \mathbf{t} \times (\nabla \times \mathbf{n}) - \mathbf{n} \times (\nabla \times \mathbf{t}).
\end{aligned}
$$

Last two terms may be expressed as

$$
\begin{aligned}
\mathbf{t} \times (\nabla \times \mathbf{n}) &= (\mathbf{n} \times \mathbf{b}) \times (\nabla \times \mathbf{n}) \\
&= \mathcal{H}_n \mathbf{b} - \mathcal{H}_{bn} \mathbf{n} \\
\mathbf{n} \times (\nabla \times \mathbf{t}) &= (\mathbf{b} \times \mathbf{t}) \times (\nabla \times \mathbf{t}) \\
&= \mathcal{H}_{bt} \mathbf{t} - \mathcal{H}_t \mathbf{b}.
\end{aligned}
$$

Collecting these for $\partial_n \mathbf{t}$ and repeating similar computations for other derivatives we obtain

$$\begin{aligned}
\partial_n \mathbf{t} &= \mathcal{H}_{bn}\mathbf{n} + \frac{1}{2}(\mathcal{H}_t - \mathcal{H}_n + \mathcal{H}_b)\mathbf{b} \\
\partial_b \mathbf{t} &= -\frac{1}{2}(\mathcal{H}_t + \mathcal{H}_n - \mathcal{H}_b)\mathbf{n} - \mathcal{H}_{nb}\mathbf{b} \\
\partial_t \mathbf{n} &= -\mathcal{H}_{bt}\mathbf{t} + \frac{1}{2}(\mathcal{H}_t - \mathcal{H}_n - \mathcal{H}_b)\mathbf{b} \\
\partial_b \mathbf{n} &= \frac{1}{2}(\mathcal{H}_t + \mathcal{H}_n - \mathcal{H}_b)\mathbf{t} + \mathcal{H}_{tb}\mathbf{b} \\
\partial_t \mathbf{b} &= \mathcal{H}_{nt}\mathbf{t} - \frac{1}{2}(\mathcal{H}_t - \mathcal{H}_n - \mathcal{H}_b)\mathbf{n} \\
\partial_n \mathbf{b} &= -\frac{1}{2}(\mathcal{H}_t - \mathcal{H}_n + \mathcal{H}_b)\mathbf{t} + \mathcal{H}_{tn}\mathbf{n}.
\end{aligned} \tag{10}$$

2.2 Constructing Frenet-Serret frame

The choice of orthonormal frame is determined by the choice of $\mathbf{n}$. In [27], we introduced such a frame assuming that the unit tangent is not an eigenvector of the curl operator. Here, we want to release this assumption and prove the existence of an orthonormal frame for all smooth dynamical systems in three dimensions. Our result will rely on the eigenvalue problem

$$\nabla \times \mathbf{t} = \lambda(\mathbf{x})\mathbf{t} \tag{11}$$

for the curl operator. If $\mathbf{t}$ is not an eigenvector of the curl operator we have $(\nabla \times \mathbf{t}) \times \mathbf{t} \neq \mathbf{0}$ and we recover the result of [27]. If however, the eigenvalue equation (11) holds, then $\mathcal{H}_t = \mathbf{t} \cdot \nabla \times \mathbf{t} = \lambda(\mathbf{x})$. At each point $\mathbf{x}$, the eigenvalue $\lambda(\mathbf{x})$ will define a surface with normal $\nabla\mathcal{H}_t(\mathbf{x})$ if $\mathcal{H}_t(\mathbf{x})$ is not a constant function. We distinguish two cases depending on whether the unit tangent has components lying on this eigensurface or not. If it has, then we choose the normal on the eigensurface. If $\mathbf{t}$ is completely aligned with the surface normal, then we recall a result of Chandrasekhar and Kendall in [28] that there exist a constant unit vector defining an eigenvector of the curl operator and construct a frame with this constant unit vector. In the remaining case with $\lambda(\mathbf{x}) = 0$, we have a surface whose gradient is the unit tangent and we choose the frame using lines of curvature of this surface. More precisely, following result proves that there are canonical liftings to Frenet-Serret frames.

Proposition 1 *Given a nonzero vector field* $\mathbf{v} \in \mathbb{R}^3$, *let* $\mathbf{t} = \mathbf{v}/||\mathbf{v}||$. *Then, the vector field* $\mathbf{n}$ *can be chosen as follows*

1. *If* $(\nabla \times \mathbf{t}) \times \mathbf{t} \neq \mathbf{0}$ *then let*

$$\mathbf{n} = \frac{(\nabla \times \mathbf{t}) \times \mathbf{t}}{||\,(\nabla \times \mathbf{t}) \times \mathbf{t}||} \tag{12}$$

and we have necessarily $\mathcal{H}_{nt} = \mathbf{n} \cdot \nabla \times \mathbf{t} = 0$.

2. *If* $(\nabla\times\mathbf{t})\times\mathbf{t}=\mathbf{0}$*, then* $\nabla\times\mathbf{t}{=}\mathcal{H}_t\mathbf{t}$ *and we have necessarily* $\mathcal{H}_{nt}=\mathcal{H}_{bt}=0$*, or equivalently* $\partial_s\mathbf{t}=\mathbf{0}$*. We distinguish two cases:*

2a. *if* $\nabla\mathcal{H}_t\times\mathbf{t}\neq\mathbf{0}$ *then choose*

$$\mathbf{n}=\frac{\nabla\mathcal{H}_t\times\mathbf{t}}{||\nabla\mathcal{H}_t\times\mathbf{t}||}. \tag{13}$$

2bi. *if* $\nabla\mathcal{H}_t\times\mathbf{t}=\mathbf{0}$ *and* $\mathcal{H}_t=$ *constant* $\neq 0$***,*** *then there exists a constant unit vector* $\mathbf{a}$ *such that*

$$\mathbf{n}=\frac{\mathbf{a}\times\mathbf{t}}{||\mathbf{a}\times\mathbf{t}||} \tag{14}$$

and hence $\mathcal{H}_{nt}=\mathcal{H}_{bt}=0$*.*

2bii. *if* $\nabla\mathcal{H}_t\times\mathbf{t}=\mathbf{0}$ *and* $\mathcal{H}_t=0$*, then there exists a surface with normal* $\mathbf{t}$ *and the Frenet-Serret frame is the Darboux frame of the lines of curvature on this surface. We have* $\mathcal{H}_{nt}=\mathcal{H}_{bt}=0$*.*

Proof. Case 1. follows easily. For case 2., assume $(\nabla\times\mathbf{t})\times\mathbf{t}=\mathbf{0}$. Then, $\nabla\times\mathbf{t}$ is proportional to $\mathbf{t}$, $\nabla\times\mathbf{t}{=}\mathcal{H}_t\mathbf{t}$ and we have $\mathcal{H}_{nt}=\mathcal{H}_{bt}=0$ which are the coefficients of unit vectors in the expression for $\partial_s\mathbf{t}$. In order to construct the normal vector $\mathbf{n}$ we have two subcases depending on whether $\nabla\mathcal{H}_t$ is non-zero and parallel to $\mathbf{t}$. If $\nabla\mathcal{H}_t\times\mathbf{t}\neq\mathbf{0}$ then define the unit normal as in Eq.(13). If $\nabla\mathcal{H}_t\times\mathbf{t}=\mathbf{0}$ we have $\nabla\mathcal{H}_t$ proportional to $\mathbf{t}$. If the proportionality function is zero, we have $\mathcal{H}_t$ =constant. In this case, $\mathbf{t}$ is an eigenvector of the curl operator with constant eigenvalue $\mathcal{H}_t$. By a result obtained in [28] there exists a scalar function ψ satisfying

$$\triangle\,\psi+\mathcal{H}_t^2\psi=0 \tag{15}$$

and a constant vector $\mathbf{a}$ such that the eigenvector $\mathbf{t}$ can be expressed as

$$\mathbf{t}=\frac{1}{\mathcal{H}_t}\nabla\times\left(\psi\mathbf{a}+\nabla\times(\psi\mathbf{a})\right). \tag{16}$$

Then, we define the normal vector by Eq. (14). It follows that

$$\begin{aligned}\mathcal{H}_{nt} &= \mathbf{n}\cdot\nabla\times\mathbf{t}=\frac{\mathbf{a}\times\mathbf{t}}{||\mathbf{a}\times\mathbf{t}||}\cdot\mathcal{H}_t\mathbf{t}=0\\ \mathcal{H}_{bt} &= \mathbf{b}\cdot\nabla\times\mathbf{t}=(\mathbf{t}\times\mathbf{n})\cdot\mathcal{H}_t\mathbf{t}=0.\end{aligned}$$

If the proportionality function is non-zero, taking curl and then dot product with $\mathbf{t}$ we get $\mathcal{H}_t=0$ which is the integrability condition for the unit tangent vector. Since, this is a subcase of 2 with $\nabla\times\mathbf{t}=\mathcal{H}_t\mathbf{t}$, we have $\nabla\times\mathbf{t}=\mathbf{0}$. Locally, there exists a function $F(\mathbf{x})$ such that $\mathbf{t}=\nabla F(\mathbf{x})$. Choose $\mathbf{n}$ to be the unit tangent vector of the line of curvature of this surface, namely, a vector $\mathbf{n}$ satisfying

$$\mathbf{n}\cdot(\mathbf{t}\times(\mathbf{n}\cdot\nabla)\,\mathbf{t})=0 \tag{17}$$

which implies, from Eq.(10) $\mathcal{H}_t=0$, $\mathcal{H}_n=\mathcal{H}_b$. Moreover, we have $\mathcal{H}_{nt}=\mathcal{H}_{bt}=0$ as in the previous case. ∎

3 Bi-Hamiltonian Structure

We shall summarize the necessary ingredients of the bi-Hamiltonian formalism in three dimensions. See [6]-[26] for details and examples. For $\mathbf{x} = \left\{x^i\right\} = (x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$ and overdot denoting the derivative with respect to t, we consider the system of autonomous differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}\left(\mathbf{x}\right) \tag{18}$$

associated with a three-dimensional smooth vector field $\mathbf{v}$. Eq.(18) is said to be Hamiltonian if the right hand side can be written as $\mathbf{v}\left(\mathbf{x}\right) = \Omega\left(\mathbf{x}\right)\left(dH\left(\mathbf{x}\right)\right)$ where $H\left(\mathbf{x}\right)$ is the Hamiltonian function and $\Omega\left(\mathbf{x}\right)$ is the Poisson bi-vector (i.e. a skew-symmetric, contravariant two-tensor) subjected to the Jacobi identity $\left[\Omega\left(\mathbf{x}\right), \Omega\left(\mathbf{x}\right)\right] = 0$ defined by the Schouten bracket [1]. In coordinates, if $\partial_i = \partial/\partial x^i$, the Poisson bi-vector is $\Omega\left(\mathbf{x}\right) = \Omega^{jk}\left(\mathbf{x}\right)\partial_j \wedge \partial_k$, with summation over repeated indices, and the Jacobi identity reads $\Omega^{i[j}\partial_i\Omega^{kl]} = 0$ where $[jkl]$ denotes the antisymmetrization over three indices. It follows that in three dimensions the Jacobi identity is a single scalar equation. One can exploit vector calculus and differential forms in three dimensions to have a more transparent understanding of Hamilton‘s equations as well as the Jacobi identity. Using the isomorphism

$$J_i = \varepsilon_{ijk}\Omega^{jk} \qquad i, j, k = 1, 2, 3 \tag{19}$$

between skew-symmetric matrices and (pseudo)-vectors defined by the completely antisymmetric Levi-Civita tensor ε_{ijk}, we can write the Hamilton's equations and the Jacobi identity as

$$\mathbf{v} = \mathbf{J} \times \nabla H \qquad \mathbf{J} \cdot \left(\nabla \times \mathbf{J}\right) = 0, \tag{20}$$

respectively. In this form the Jacobi identity is equivalent to the Frobenius integrability condition $J \wedge dJ = 0$ for the one form $J = J_i dx^i$. It is the condition for J to define a foliation of codimension one in three dimensional space [17],[29]-[31]. A distinguished property of Poisson structures in three dimensions is the invariance of the Jacobi identity under the multiplication of Poisson vector $\mathbf{J}\left(\mathbf{x}\right)$ by an arbitrary but non-zero factor. The identities

$$\mathbf{J} \cdot \mathbf{v} = 0, \quad \nabla H \cdot \mathbf{v} = 0 \tag{21}$$

follows directly from the Hamilton's equations in (20). The second equation in (21) is the expression for the conservation of Hamiltonian function. A three dimensional vector field $\mathbf{v}\left(\mathbf{x}\right)$ is said to be bi-Hamiltonian if there exist two different compatible Hamiltonian structures [3],[32]. In the notation of equation (20), this implies

$$\mathbf{v} = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1 \tag{22}$$

for the dynamical equations. The compatibility condition for $\mathbf{J}_1$ and $\mathbf{J}_2$ is defined by the Jacobi identity for the Poisson pencil $\mathbf{J}_1 + c\mathbf{J}_2$ for arbitrary constant c.

3.1 Jacobi identity in Frenet-Serret frame

Given a vector field $\mathbf{v}$, $\mathbf{v}/\left\|\mathbf{v}\right\|$ is the unit tangent vector $\mathbf{t}$ to the flow of $\mathbf{v}$. It follows from the identity $\mathbf{J}\cdot\mathbf{v}=0$ that the Poisson vector $\mathbf{J}$ has no component along the unit tangent vector $\mathbf{t}$. Hence, we set

$$\mathbf{J}=A\mathbf{n}+B\mathbf{b} \tag{23}$$

for unknown functions $A\left(\mathbf{x}\right)$ and $B\left(\mathbf{x}\right)$ satisfying $A^2+B^2\neq 0$. Assuming $A\neq 0$ and defining the function $\mu=B/A$ the Jacobi identity for $\mathbf{J}=A(\mathbf{n}+\mu\mathbf{b})$ reduces to the Riccati equation

$$\partial_s\mu=\mathcal{H}_n+\mu(\mathcal{H}_{nb}+\mathcal{H}_{bn})+\mu^2\mathcal{H}_b \tag{24}$$

and

$$\partial_s\ln A=\partial_s\ln\left\|\mathbf{v}\right\|-\mu\mathcal{H}_b-\mathcal{H}_{nb} \tag{25}$$

in the arclenght variable s. The Riccati equation (24) is equivalent to a linear second order equation and hence possesses two linearly independent solutions leading to two Poisson vectors for dynamical system under consideration. The Hamiltonian form of dynamical equations implies that the Poisson vectors obtained from solutions of Riccati equation are always compatible. Thus, we conclude that

Proposition 2 *All dynamical systems in three dimensions possess two compatible Poisson vectors.*

3.2 Poisson vectors and conserved quantities

Once we have the independent solutions μ_1 and μ_2 of the Riccati equation, we can form the compatible Poisson vectors

$$\mathbf{J}_1{=}A_1(\mathbf{n}+\mu_1\mathbf{b}),\qquad \mathbf{J}_2{=}A_2(\mathbf{n}+\mu_2\mathbf{b}) \tag{26}$$

with conformal factors A_1 and A_2. The construction of corresponding Hamiltonian functions to form a bi-Hamiltonian pair requires integration of these Poisson vectors.

Proposition 3 *The conserved covariants for* $\mathbf{J}_1$ *and* $\mathbf{J}_2$ *are*

$$\nabla H_1=\frac{||\mathbf{v}||}{A_1A_2(\mu_2-\mu_1)}\mathbf{J}_1,\quad \nabla H_2=\frac{-||\mathbf{v}||}{A_1A_2(\mu_2-\mu_1)}\mathbf{J}_2, \tag{27}$$

respectively.

Proof. By definition, the conserved Hamiltonians satisfy $\mathbf{v}\cdot\nabla H_1=\mathbf{v}\cdot\nabla H_1=0$. That means, the gradients lie on the space spanned by $\{\mathbf{n}_1,\mathbf{n}_2\}$ or, equivalently, by $\{\mathbf{J}_1,\mathbf{J}_2\}$. We, thus, form the linear combinations

$$\nabla H_1=a\mathbf{J}_1{+}b\mathbf{J}_2,\quad \nabla H_2=c\mathbf{J}_1{+}d\mathbf{J}_2$$

and determine the coefficients. Bi-Hamiltonian form in Eq.(22) together with the orthonormality of the basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ imply

$$b = -c = \frac{||\mathbf{v}||}{A_1 A_2(\mu_2 - \mu_1)}.$$

The identities $\nabla \times \nabla H_1 = \nabla \times \nabla H_2 \equiv 0$ dotted with $\mathbf{J}_1$ and $\mathbf{J}_2$ result, recalling also the compatibility condition, in the fact that a and d are also multiples of this function. It turns out that, for each Hamiltonian, there is an equivalence class of Hamiltonian functions whose gradients differ by the functions a and d. Hence, without restriction to generality, the conserved covariants are given by Eq.(27). ■

4 Gradient Systems

We assume that there exists a potential function F for the velocity field

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}) = \nabla F(\mathbf{x}) \tag{28}$$

of a given dynamical system in three dimensions. In this case, we have $\nabla \times \mathbf{v}(\mathbf{x}) = \mathbf{0}$, so that

$$\nabla \times \mathbf{t} = \mathbf{t} \times \nabla \ln \|\mathbf{v}\| \tag{29}$$

and hence, $\mathbf{t} \cdot \nabla \times \mathbf{t} = 0$. Then, either $(\nabla \times \mathbf{t}) \times \mathbf{t} \neq \mathbf{0}$ or $\nabla \times \mathbf{t} = \mathbf{0}$. In the first case we take $\mathbf{n} = ((\nabla \times \mathbf{t}) \times \mathbf{t})/||(\nabla \times \mathbf{t}) \times \mathbf{t}||$ with $\mathbf{n} \cdot \nabla \times \mathbf{t} = 0$ as well. Thus, the Frenet-Serret frame for a gradient system with potential function F may consists of unit vector fields

$$\mathbf{t} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\nabla F}{||\nabla F||}, \quad \mathbf{n} = \frac{(\nabla \times \mathbf{t}) \times \mathbf{t}}{||\nabla \times \mathbf{t}||}, \quad \mathbf{b} = \frac{\nabla \times \mathbf{t}}{||\nabla \times \mathbf{t}||}. \tag{30}$$

In the second case, there exists a surface with normal $\mathbf{t}$, which is nothing but any potential surface of $F(\mathbf{x})$ and the Frenet-Serret frame is the Darboux frame of lines of curvature on the potential surface. In other words, $\mathbf{t}$ will be unit normal of the potential surfaces and choosing the Frenet-Serret frame as the Darboux frame of the lines of curvature on the potential surface will work in both cases.

Now, since $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms an orthonormal frame, the arclengths of their integral curves (s,n,b) form a local coordinate system around any point $p \in \mathbb{R}^3$ provided that $\mathbf{v}(p) \neq \mathbf{0}$. On the other hand,

$$||\mathbf{v}||\mathbf{t} = \nabla F(\mathbf{x}) \tag{31}$$

implies that

$$\begin{array}{rcccc} \mathbf{t}\cdot\nabla F(\mathbf{x}) & = & \partial_s F(s,n,b) & = & ||\mathbf{v}|| \\ \mathbf{n}\cdot\nabla F(\mathbf{x}) & = & \partial_n F(s,n,b) & = & 0 \\ \mathbf{b}\cdot\nabla F(\mathbf{x}) & = & \partial_b F(s,n,b) & = & 0 \end{array} \tag{32}$$

Hence $F(\mathbf{x}) = F(s)$ in the coordinate system defined above. Therefore, with the assumption that $\mathbf{v}(p) = \nabla F(p) \neq \mathbf{0}$ the potential surfaces are

$$F(\mathbf{x}) = F(s) = c' \tag{33}$$

which implies $s = c$. As s is the arclength of integral curve of (28), its flow Φ_σ acts on the s coordinate and therefore on the potential surfaces by translation, i.e. $\Phi_\sigma(s,n,b) = (s+\sigma,n,b)$. If $H(\mathbf{x}) = H(s,n,b)$ is a Hamiltonian function for (28), then it will be invariant by this flow $H(s+\sigma,n,b) = H(s,n,b)$, which implies that H is independent of s, in other words fixing $s = c$ and letting $\sigma = s$ lead to

$$H(s,n,b) = H(s,n,b)\,|_{s=c}\ . \tag{34}$$

The invariance of Hamiltonians under the flow implies that it is sufficient to determine the Hamiltonian function on any potential surface. Namely, a Hamiltonian function can be reduced to a Hamiltonian function on potential surfaces. However, the converse may not be so straightforward. For the construction of a Hamiltonian function out of a function defined on potential surfaces, or equivalently, to extend a gradient vector ∇H on the potential surface to a gradient vector on $\mathbb{R}^3$, the vectors $\mathbf{t}$ and $\mathbf{t}\times\nabla H$, which are perpendicular to ∇H, must be tangent to a surface in $\mathbb{R}^3$. This condition can be written as

$$[\alpha\mathbf{t}\cdot\nabla, (\beta\mathbf{t}\times\nabla H)\cdot\nabla] = 0 \tag{35}$$

for some functions α and β on $\mathbb{R}^3$, where $[\cdot,\cdot]$ denotes the bracket of vector fields. Our purpose is to prove that

Theorem 4 *The Hamiltonian functions of a gradient system defined by a potential function F are determined by geodesic distance functions defined by non-conjugate points on the potential surfaces.*

To obtain this result, first we are going to show that, geodesic distances on a potential surface are defined by a gradient system. Then, we will prove that all Hamiltonian functions on potential surface are generated by geodesic distances to non-conjugate points. Finally, we will show that these distance functions can be extended to Hamiltonian functions on $\mathbb{R}^3$.

4.1 Differential geometry of potential surfaces

We give geometric parameters of potential surface and of an arbitrary curve on it.

Proposition 5 *The fundamental forms of surfaces $F(s) = c$ are*

$$\begin{aligned} ds_1^2 &= dn^2 + db^2 \\ ds_2^2 &= \mathcal{H}_{bn}dn^2 + (\mathcal{H}_b - \mathcal{H}_n)dbdn - \mathcal{H}_{nb}db^2. \end{aligned}$$

If $\mathbf{X}\left(n\left(\sigma\right),b\left(\sigma\right)\right)$ *is a curve, parametrized with the arclenght* σ*, on the potential surface, then the normal and the geodesic curvatures are*

$$\begin{aligned} \kappa_n &= \xi^2\mathcal{H}_{bn}+\xi\eta(\mathcal{H}_n-\mathcal{H}_b)+\eta^2\mathcal{H}_{nb} && (36)\\ \kappa_g &= \partial_b\xi-\partial_n\eta-\xi\mathcal{H}_{tn}-\eta\mathcal{H}_{tb}, && (37)\end{aligned}$$

where we denote

$$dn/d\sigma=\xi, \quad db/d\sigma=\eta.$$

Proof. Let $\mathbf{X}=\mathbf{X}(c,n,b)$ be a potential surface of the function F determined by a constant c. The tangent vectors and the unit normal vector of this potential surface are

$$\partial_n\mathbf{X}=\mathbf{n}, \quad \partial_b\mathbf{X}=\mathbf{b}, \quad \partial_n\mathbf{X}\times\ \partial_b\mathbf{X}=\mathbf{t}.$$

The first fundamental form is obviously $ds_1^2=dn^2+db^2$. To find the second fundamental form, first note that the normal vector $\mathbf{t}(c,n,b)$ of $\mathbf{X}(c,n,b)$ is independent of s. We compute

$$\begin{aligned} d\mathbf{t}\cdot d\mathbf{X} &= (\partial_n\mathbf{t}dn+\partial_b\mathbf{t}db)\cdot(\mathbf{n}dn+\mathbf{b}db)\\ &= \mathbf{n}\cdot(\partial_n\mathbf{t}dn^2+\partial_b\mathbf{t}dbdn)+\mathbf{b}\cdot(\partial_n\mathbf{t}dndb+\partial_b\mathbf{t}db^2)\\ &= (\mathbf{n}\cdot\partial_b\mathbf{t}+\mathbf{b}\cdot\partial_n\mathbf{t})dbdn+\mathbf{n}\cdot\partial_n\mathbf{t}dn^2+\mathbf{b}\cdot\partial_b\mathbf{t}db^2\\ &= \mathcal{H}_{bn}dn^2+(\mathcal{H}_b-\mathcal{H}_n)dbdn-\mathcal{H}_{nb}db^2\end{aligned}$$

and thereby obtaining the second fundamental form. For curvatures, the unit tangent vector of the curve $\mathbf{X}\left(n\left(\sigma\right),b\left(\sigma\right)\right)$ is

$$\mathbf{T}=\frac{d\mathbf{X}}{d\sigma}=\frac{dn}{d\sigma}\mathbf{n}+\frac{db}{d\sigma}\mathbf{b} \qquad (38)$$

with $(dn/d\sigma)^2+(db/d\sigma)^2=1$. We compute the curvature vector

$$\begin{aligned} \frac{d\mathbf{T}}{d\sigma} &= \frac{d\xi}{d\sigma}\mathbf{n}+\xi\frac{d\mathbf{n}}{d\sigma}+\frac{d\eta}{d\sigma}\mathbf{b}+\eta\frac{d\mathbf{b}}{d\sigma}=\kappa\mathbf{N}\\ &= (\xi\partial_n\xi+\eta\partial_b\xi)\mathbf{n}+\xi(\xi\partial_n\mathbf{n}+\eta\partial_b\mathbf{n})\\ &\quad +(\xi\partial_n\eta+\eta\partial_b\eta)\mathbf{b}+\eta(\xi\partial_n\mathbf{b}+\eta\partial_b\mathbf{b}).\end{aligned}$$

The second and the third parenthesis can be expressed, using formulas from calculus in Frenet-Serret frame, as

$$\begin{aligned} \xi\partial_n\mathbf{n}+\eta\partial_b\mathbf{n} &= \xi(\mathcal{H}_{tn}\mathbf{b}-\mathcal{H}_{bn}\mathbf{t})+\eta(-\mathcal{H}_{tn}\mathbf{n}+\frac{1}{2}(\mathcal{H}_n-\mathcal{H}_b)\mathbf{t}),\\ \xi\partial_n\mathbf{b}+\eta\partial_b\mathbf{b} &= \eta(-\mathcal{H}_{tb}\mathbf{n}+\mathcal{H}_{nb}\mathbf{t})+\xi(\mathcal{H}_{tb}\mathbf{b}+\frac{1}{2}(\mathcal{H}_n-\mathcal{H}_b)\mathbf{t}),\end{aligned}$$

with which the curvature vector becomes

$$\begin{aligned} \frac{d\mathbf{T}}{d\sigma} &= (-\xi^2\mathcal{H}_{bn}+\xi\eta(\mathcal{H}_n-\mathcal{H}_b)+\eta^2\mathcal{H}_{nb})\mathbf{t}\\ &\quad +(-\eta\partial_n\eta+\eta\partial_b\xi-\xi\eta\mathcal{H}_{tn}-\eta^2\mathcal{H}_{tb})\mathbf{n}\\ &\quad +(\xi\partial_n\eta-\xi\partial_b\xi+\xi^2\mathcal{H}_{tn}+\xi\eta\mathcal{H}_{tb})\mathbf{b}\\ &= (-\xi^2\mathcal{H}_{bn}+\xi\eta(\mathcal{H}_n-\mathcal{H}_b)+\eta^2\mathcal{H}_{nb})\mathbf{t}\\ &\quad +(\partial_b\xi-\partial_n\eta-\xi\mathcal{H}_{tn}-\eta\mathcal{H}_{tb})(\eta\mathbf{n}-\xi\mathbf{b})\end{aligned}$$

where we used $\xi\partial_n\xi = -\eta\partial_n\eta$ and $\eta\partial_b\eta = -\xi\partial_b\xi$. This gives the decomposition

$$\frac{d\mathbf{T}}{d\sigma} = \kappa_n\mathbf{t} + \kappa_g(\eta\mathbf{n} - \xi\mathbf{b})$$

of the curvature into tangential and geodesic components. ■

A curve parametrized with arclength is a geodesic on a surface if $\kappa_g = 0$ for all points on the curve. We will see that this condition is necessary for the integrability of forms forming transformations to geodesic coordinates on the potential surface of a dynamical system. There will be two such coordinate systems, each corresponds to construction of a Hamiltonian function on the surface.

4.2 Flows of Hamiltonian functions on potential surfaces and geodesic distances

In this section, we shall prove two propositions which relate the flows of Hamiltonian functions on potential surfaces with geodesic distances. First proposition tells that geodesics are integral curves of gradient flows of geodesic distances on potential surfaces. Second proposition proves that geodesic distances defined by non-conjugate points, i.e. points which are joined by a unique geodesic, are functionally independent and therefore defines two Hamiltonian functions. Hence these functions generates all Hamiltonian functions that can be defined on the potential surface.

Proposition 6 *Finding two Hamiltonian functions for the gradient system ∇F on the potential surface $F = c$ amounts to finding geodesic distances on the potential surface.*

Proof. Let (u, v) be orthogonal coordinates on the potential surface $F(\mathbf{x}) = c$ for a given Riemannian metric

$$(g_{ij}(u,v)) = \begin{pmatrix} g_{uu}(u,v) & 0 \\ 0 & g_{vv}(u,v) \end{pmatrix}$$

and let (q, p) be another orthogonal coordinate system such that

$$(G_{ij}(q,p)) = \begin{pmatrix} G_{qq}(q,p) & 0 \\ 0 & G_{pp}(q,p) \end{pmatrix}.$$

Above two metrics are related by

$$\begin{aligned} g_{uu}(u,v) &= G_{qq}(q,p)\, q_u^2 + G_{pp}(q,p)\, p_u^2 \\ g_{vv}(u,v) &= G_{qq}(q,p)\, q_v^2 + G_{pp}(q,p)\, p_v^2 \\ 0 &= G_{qq}(q,p)\, q_u q_v + G_{pp}(q,p)\, p_u p_v \end{aligned} \tag{39}$$

Note that, although (u, v) and (q, p) are orthogonal coordinates on the surface, the coordinate transformation between them need not be orthogonal, that is,

the Jacobian determinant

$$J = q_u p_v - q_v p_u = \sqrt{\frac{g_{uu} g_{vv}}{G_{qq} G_{pp}}} \tag{40}$$

is not necessarily equal to unity. Defining

$$\gamma_u^q = \sqrt{\frac{g_{uu}}{G_{qq}}}, \quad \gamma_v^q = \sqrt{\frac{g_{vv}}{G_{qq}}}, \quad \gamma_u^p = \sqrt{\frac{g_{uu}}{G_{pp}}}, \quad \gamma_v^p = \sqrt{\frac{g_{vv}}{G_{pp}}} \tag{41}$$

and introducing a parameter θ, we can write equations (39) as a system of differential equations

$$\begin{aligned} \frac{\partial q}{\partial u} &= \gamma_u^q \cos\theta, \qquad \frac{\partial p}{\partial u} = \gamma_u^p \sin\theta && (42) \\ \frac{\partial q}{\partial v} &= -\gamma_v^q \sin\theta, \qquad \frac{\partial p}{\partial v} = \gamma_v^p \cos\theta. && (43) \end{aligned}$$

for transformations between two orthogonal coordinates. Equations (42) and (43) are subjected to two integrability conditions

$$\begin{aligned} \gamma_u^q \sin\theta \frac{\partial\theta}{\partial v} - \gamma_v^q \cos\theta \frac{\partial\theta}{\partial u} &= \frac{\partial\gamma_u^q}{\partial v}\cos\theta + \frac{\partial\gamma_v^q}{\partial u}\sin\theta && (44) \\ \gamma_u^p \cos\theta \frac{\partial\theta}{\partial v} + \gamma_v^p \sin\theta \frac{\partial\theta}{\partial u} &= \frac{\partial\gamma_v^p}{\partial u}\cos\theta - \frac{\partial\gamma_u^p}{\partial v}\sin\theta. && (45) \end{aligned}$$

The characteristic curve of the first integrability condition in Eq.(44) is the integral curve of the dynamical system defined by

$$\begin{aligned} \frac{du}{dt} &= -\gamma_v^q \cos\theta \\ \frac{dv}{dt} &= \gamma_u^q \sin\theta \\ \frac{d\theta}{dt} &= \frac{\partial\gamma_u^q}{\partial v}\cos\theta + \frac{\partial\gamma_v^q}{\partial u}\sin\theta \end{aligned} \tag{46}$$

and the arclength σ_1 of this integral curve satisfies

$$\frac{d\sigma_1}{dt} = \sqrt{g_{vv}}\gamma_u^q.$$

In the transformed coordinates (q,p), the first two equations for the characteristic curve become

$$\frac{dq}{dt} = -\frac{1}{\sqrt{G_{qq}}}\frac{d\sigma_1}{dt}, \qquad \frac{dp}{dt} = 0 \tag{47}$$

hence, the arclength ρ_1 in the transformed coordinates satisfies

$$\frac{d\rho_1}{dt} = \frac{d\sigma_1}{dt}$$

and Eq.(47) in arclength parametrization becomes

$$\frac{d}{d\rho_1}\begin{pmatrix} q \\ p \end{pmatrix} = -\sqrt{G_{qq}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{48}$$

The geodesic curvature of the characteristic curve $q = c$ is given by

$$\kappa_g = \frac{1}{2\sqrt{G_{pp}}}\frac{\partial \ln G_{qq}}{\partial q} \tag{49}$$

Similarly, for the second integrability condition (45), the characteristic equation is the integral curve of the dynamical system

$$\begin{aligned} \frac{du}{dt} &= \gamma_v^p \sin\theta \\ \frac{dv}{dt} &= \gamma_u^p \cos\theta \\ \frac{d\theta}{dt} &= \frac{\partial \gamma_v^p}{\partial u}\cos\theta - \frac{\partial \gamma_u^p}{\partial v}\sin\theta \end{aligned} \tag{50}$$

and the arclength σ_2 of this curve is determined by

$$\frac{d\sigma_2}{dt} = \sqrt{g_{uu}}\gamma_v^p.$$

In the transformed coordinates (q, p), the first two equations for characteristic curve are

$$\frac{dq}{dt} = 0, \qquad \frac{dp}{dt} = \frac{1}{\sqrt{G_{pp}}}\frac{d\sigma_2}{dt} \tag{51}$$

and the arclength ρ_2 in the transformed coordinates satisfies

$$\frac{d\rho_2}{dt} = \frac{d\sigma_2}{dt}. \tag{52}$$

Equation (51) in this parametrization becomes

$$\frac{d}{d\rho_2}\begin{pmatrix} q \\ p \end{pmatrix} = -\sqrt{G_{pp}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{53}$$

The geodesic curvature of the characteristic curve $p = c$ is

$$\kappa_g = \frac{1}{2\sqrt{G_{qq}}}\frac{\partial \ln G_{pp}}{\partial p}. \tag{54}$$

To arrive at the conclusion of the proposition we will restrict the characteristic curves to be geodesics. To this end, we note that we can choose two types of geodesic coordinates on a surface. One transforming the metric as

$$\begin{pmatrix} g_{uu}(u,v) & 0 \\ 0 & g_{vv}(u,v) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & G_{pp}(q,p) \end{pmatrix} \tag{55}$$

whereas, the other transforming the metric as

$$\begin{pmatrix} g_{uu}(u,v) & 0 \\ 0 & g_{vv}(u,v) \end{pmatrix} \longrightarrow \begin{pmatrix} G_{qq}(q,p) & 0 \\ 0 & 1 \end{pmatrix}. \tag{56}$$

The first transformation reduces the equations (48) and (49) to

$$\frac{d}{d\rho_1}\begin{pmatrix} q \\ p \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \kappa_g = 0 \tag{57}$$

for the characteristic curve of the first integrability condition in Eq.(44) and its geodesic curvature. Similarly, the second transformation reduces the equations (53) and (54) to

$$\frac{d}{d\rho_2}\begin{pmatrix} q \\ p \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \kappa_g = 0 \tag{58}$$

which are associated to characteristic curve of the second integrability condition in Eq.(45).

From Eq.(47) and Eq.(51) we have

$$\frac{dq}{dt} = -\frac{d\sigma_1}{dt}, \quad \frac{dp}{dt} = \frac{d\sigma_2}{dt}$$

and without restriction of generality, we may assume that

$$H_1 = -q = \sigma_1 \quad H_2 = p = \sigma_2 \tag{59}$$

which are arclengths along geodesic curves, namely geodesic distances. ■

Next, we are going to prove that geodesic distance functions defined by non-conjugate points are two Hamiltonian functions on the potential surfaces.

Proposition 7 *The Hamiltonian functions of a gradient system defined by a potential function F are determined by geodesic distance functions on potential surfaces.*

Proof. Since level surfaces, $F(\mathbf{x}) = c$, are closed subspaces of $\mathbb{R}^3$, which is a complete metric space, level surfaces with the induced metric are also complete. Then, by Hopf-Rinow theorem [37] they are also geodesically complete, which implies the existence of a geodesic between any two points on a level surface. Choose and fix a level surface $F(\mathbf{x}) = c$, which we denote by S_c, and a point $\mathbf{p}_1 \in S_c$. Then the first Hamiltonian function on the potential surface, $H_1(\mathbf{x})$, can be constructed as

$$H_1(\mathbf{x}) = d(\mathbf{x}, \mathbf{p}_1) = \text{The length of the geodesic joining } \mathbf{x} \in S_c \text{ to } \mathbf{p}_1.$$

Since the gradients of distance functions have unit norm,

$$\|\nabla H_1(\mathbf{x})\| = \|\nabla d(\mathbf{x}, \mathbf{p}_1)\| = 1 \text{ for } \mathbf{x} \in S_c$$

and belong to tangent plane of S_c at $\mathbf{x}$, it is a Hamiltonian function of (28), whose gradient has unit norm on the potential surface.

For the second Hamiltonian, let $\Gamma(\mathbf{p}_1)$ be the set of intersection of all geodesics through $\mathbf{p}_1$. Choosing another point $\mathbf{p}_2 \notin \Gamma$ on the level surface S_c, and repeating the same construction, the second Hamiltonian function is obtained. This second function will be independent from the first one. In the rest of the proof we will show this. Assume, on the contrary, that the second distance function is not independent. Then,

$$\nabla d(\mathbf{x}, \mathbf{p}_1) \times \nabla d(\mathbf{x}, \mathbf{p}_2) = 0$$

which implies that

$$\nabla d(\mathbf{x}, \mathbf{p}_2) = \lambda \nabla d(\mathbf{x}, \mathbf{p}_1)$$

and since they have unit norm

$$\|\nabla d(\mathbf{x}, \mathbf{p}_1)\| = \|\nabla d(\mathbf{x}, \mathbf{p}_2)\| = 1$$

we obtain $\lambda = \pm 1$. Now, if $\lambda = 1$ then,

$$\nabla d(\mathbf{x}, \mathbf{p}_1) = \nabla d(\mathbf{x}, \mathbf{p}_2)$$

and therefore

$$d(\mathbf{x}, \mathbf{p}_2) = d(\mathbf{x}, \mathbf{p}_1) + k.$$

Choosing $\mathbf{x}$ as the midpoint of the geodesic combining $\mathbf{p}_1$ and $\mathbf{p}_2$ implies that $k = 0$ on the surface, and therefore $\mathbf{p}_2 = \mathbf{p}_1 \in \Gamma(\mathbf{p}_1)$ which contradicts our assumption. On the other hand if $\lambda = -1$, then

$$\nabla d(\mathbf{x}, \mathbf{p}_1) = -\nabla d(\mathbf{x}, \mathbf{p}_2)$$

and therefore

$$d(\mathbf{x}, \mathbf{p}_1) + d(\mathbf{x}, \mathbf{p}_2) = k$$

Setting $\mathbf{x} = \mathbf{p}_1$ implies that $k = d(\mathbf{p}_1, \mathbf{p}_2)$, and the condition becomes

$$d(\mathbf{x}, \mathbf{p}_1) + d(\mathbf{x}, \mathbf{p}_2) = d(\mathbf{p}_1, \mathbf{p}_2).$$

By triangle inequality, this means that the points $\mathbf{x}, \mathbf{p}_1, \mathbf{p}_2$ lie on the same geodesic for all $\mathbf{x}$ on the level surface. Hence, $\nabla d(\mathbf{x}, \mathbf{p}_1) = -\nabla d(\mathbf{x}, \mathbf{p}_2)$ is possible only if every geodesic through $\mathbf{p}_1$ contains $\mathbf{p}_2$, and therefore $\mathbf{p}_2 \in \Gamma(\mathbf{p}_1)$ which again contradicts to our assumption. Namely, distance functions defined by these two points specified above are functionally independent. Since the gradient of these two functions span the tangent plane at any point, gradient of any third function obtained in this way will be linearly dependent on the previous two, therefore it will be a function of H_1 and H_2. ■

To sum up, if we take $\mathbf{X}(n(\sigma_i), b(\sigma_i))$ to be the gradient flow for $i - th$ Hamiltonian function on the potential surface, we get

$$\xi = \frac{dn}{d\sigma} = \partial_n H_i, \quad \eta = \frac{db}{d\sigma} = \partial_b H_i, \qquad i = 1, 2.$$

together with

$$\|\nabla H_i\|^2 = (\partial_n H_i)^2 + (\partial_b H_i)^2 = 1$$

Furthermore, from Eqs.(26) and (27) we can identify the partial derivatives of Hamiltonian function and obtain

$$\partial_b H_i = \mu_i \partial_n H_i \tag{60}$$

or $\eta = \mu_i \xi$ and $\eta^2 + \xi^2 = 1$. Then, the gradient of Hamiltonian functions on potential surface become two different representations of the unit tangent vector of $\mathbf{X}(n(\sigma), b(\sigma))$

$$\mathbf{T} = \nabla H_i = \frac{1}{\sqrt{1+\mu_i^2}} (\mathbf{n} + \mu_i \mathbf{b}) = \frac{\mathbf{J}_i}{\|\mathbf{J}_i\|}, \qquad i = 1, 2. \tag{61}$$

The integrability condition

$$\partial_n (\partial_b H_i) - \partial_b (\partial_n H_i) + \mathcal{H}_{tn} \partial_n H + \mathcal{H}_{tb} \partial_b H = 0 \tag{62}$$

of Eq.(60)

$$\kappa_g = \frac{1}{(1+\mu^2)^{\frac{3}{2}}} \left(\partial_n \mu + \mu \partial_b \mu + \left(1+\mu^2\right) (\mathcal{H}_{tn} + \mu \mathcal{H}_{tb}) \right) = 0 \tag{63}$$

is the vanishing geodesic curvature described in Eq.(37).

Note that the Riccati equation (24) describing $\mu(s, n, b)$ determines the partial derivative with respect to s variable and allows an arbitrary dependence on n and b variables, while the vanishing geodesic curvature condition depends only on the derivatives with respect to n and b. Therefore, the choice of geodesic distances as Hamiltonian functions puts a restriction on the arbitrariness admitted by Eq.(24).

4.3 Local extension of Hamiltonian functions on potential surfaces

On any potential surface, $F(\mathbf{x}) = c$, we had two gradient vectors $\nabla H_1, \nabla H_2$ and the unit normal of the surface $\mathbf{t}$. One can extend these Hamiltonian functions on potential surfaces to functions on $\mathbb{R}^3$ in infinitely many ways. Geometrically this amounts to embedding geodesic curves to surfaces in $\mathbb{R}^3$, which are the potential surfaces of Hamiltonian functions of dynamical system (28) under consideration. These surfaces must contain integral curves of Eq.(28). In other words, tangent plane of surface obtained by extension of ∇H_i must contain all vectors perpendicular to each ∇H_i in $\mathbb{R}^3$, in particular, $\mathbf{t}$ and $\nabla H_i \times \mathbf{t}$. In fact, since $\mathbf{t}$ and $\nabla H_i \times \mathbf{t}$ are linearly independent by definition, they span a two dimensional plane. The following proposition proves that they always integrate to potential surface of the corresponding Hamiltonian function. For convenience we will assume that Eq.(28) is of the form

$$\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla} F(\mathbf{x}) = \mathbf{J}_1(\mathbf{x}) \times \nabla H_2(\mathbf{x}) = \mathbf{J}_2(\mathbf{x}) \times \nabla H_1(\mathbf{x}) \tag{64}$$

of a bi-Hamiltonian system, where

$$\mathbf{J}_1 = \phi \nabla H_1, \quad \mathbf{J}_2 = -\phi \nabla H_2 \tag{65}$$

and the conformal factor is given by

$$\phi = \frac{A_1 A_2 (\mu_2 - \mu_1)}{||\mathbf{v}||}. \tag{66}$$

Proposition 8

$$\left[\frac{1}{||\mathbf{v}||} \mathbf{t} \cdot \nabla, \frac{1}{||\mathbf{v}||} \left(\mathbf{J}_i \times \mathbf{t} \right) \cdot \nabla \right] = \mathbf{0}, \qquad i = 1, 2 \tag{67}$$

Proof. Let $\mathbf{w} = \xi \mathbf{n} + \eta \mathbf{b}$ be a vector field in the tangent plane of a potential surface of the function F. Then

$$\begin{aligned} [\mathbf{t} \cdot \nabla, \mathbf{w} \cdot \nabla] \quad = \quad & -\mathcal{H}_{bt} \xi \mathbf{t} \cdot \nabla + \left(\partial_s \xi - \mathcal{H}_{bn} \xi + \mathcal{H}_n \eta \right) \mathbf{n} \cdot \nabla \\ & + \left(\partial_s \eta - \mathcal{H}_b \xi + \mathcal{H}_{nb} \eta \right) \mathbf{b} \cdot \nabla. \end{aligned} \tag{68}$$

If $\mathbf{w} = \nabla H$ for some function H on the potential surface, we get

$$\begin{aligned} [\mathbf{t} \cdot \nabla, \nabla H \cdot \nabla] \quad = \quad & -\mathcal{H}_{bt} \partial_n H \mathbf{t} \cdot \nabla - \left(\mathbf{2} \mathcal{H}_{bn} \partial_n H + (\mathcal{H}_b - \mathcal{H}_n) \partial_b H \right) \mathbf{n} \cdot \nabla \\ & + \left(2 \mathcal{H}_{nb} \partial_b H + \left(\mathcal{H}_n - \mathcal{H}_b \right) \partial_n H \right) \mathbf{b} \cdot \nabla. \end{aligned} \tag{69}$$

Using (69), it is easy to compute

$$[\mathbf{t} \cdot \nabla, (\nabla H \times \mathbf{t}) \cdot \nabla] = -\mathcal{H}_{bt} \partial_b H \mathbf{t} \cdot \nabla - (\nabla \cdot \mathbf{t}) \left(\nabla H \times \mathbf{t} \right) \cdot \nabla \tag{70}$$

which is sufficient to show that the space spanned by $\mathbf{t}$ and $\nabla H \times \mathbf{t}$ is tangent to a surface in $\mathbb{R}^3$. Using the equations

$$\partial_s \mu_i = \mathcal{H}_n + \mu_i (\mathcal{H}_{nb} + \mathcal{H}_{bn}) + \mu_i^2 \mathcal{H}_b \qquad i = 1, 2 \tag{71}$$

$$\partial_s \ln A_i = \partial_s \ln \|\mathbf{v}\| - \mu_i \mathcal{H}_b - \mathcal{H}_{nb} \qquad i = 1, 2 \tag{72}$$

defining the Poisson structures we obtain

$$\partial_s \ln \left(\mu_2 - \mu_1 \right) = \left(\mathcal{H}_{nb} + \mathcal{H}_{bn} \right) + \left(\mu_2 + \mu_1 \right) \mathcal{H}_b \tag{73}$$

$$\partial_s \ln \frac{A_i}{\|\mathbf{v}\|} = -\mu_i \mathcal{H}_b - \mathcal{H}_{nb}. \tag{74}$$

Eqs.(73) and (74) when used in Eq.(66) result in

$$\partial_s \ln \frac{\phi}{||\mathbf{v}||} = \mathcal{H}_{bn} - \mathcal{H}_{nb} = \nabla \cdot \mathbf{t} \tag{75}$$

for the divergence of $\mathbf{t}$. This leads, with Eq.(70), to

$$\left[\mathbf{t} \cdot \nabla, \frac{\phi}{||\mathbf{v}||} \left(\nabla H_i \times \mathbf{t} \right) \cdot \nabla \right] = -\frac{\phi}{||\mathbf{v}||} \mathcal{H}_{bt} \partial_b H_i \mathbf{t} \cdot \nabla \tag{76}$$

By Eq.(74) we have

$$\partial_n H_1 = -\frac{A_1}{\phi}, \quad \partial_b H_1 = -\frac{A_1\mu_1}{\phi}, \quad \partial_n H_2 = \frac{A_2}{\phi}, \quad \partial_b H_1 = \frac{A_2\mu_2}{\phi} \tag{77}$$

and we can rewrite (76) in the following form:

$$\left[\mathbf{t}\cdot\nabla, \frac{A_i}{||\mathbf{v}||}\left(\mu_i \mathbf{n} - \mathbf{b}\right)\cdot\nabla\right] = -\frac{A_i\mu_i}{||\mathbf{v}||}\mathcal{H}_{bt}\mathbf{t}\cdot\nabla \tag{78}$$

Now, for any vector field $\mathbf{u}$ satisfying

$$[\mathbf{t}\cdot\nabla, \mathbf{u}\cdot\nabla] = \lambda\mathbf{t}\cdot\nabla \tag{79}$$

we can find an integrating factor α such that

$$[\alpha\mathbf{t}\cdot\nabla, \mathbf{u}\cdot\nabla] = \left(\alpha\lambda - \mathbf{u}\cdot\nabla\alpha\right)\mathbf{t}\cdot\nabla = \mathbf{0} \tag{80}$$

and therefore

$$\alpha\lambda - \mathbf{u}\cdot\nabla\alpha = 0 \tag{81}$$

Taking

$$\mathbf{u} = \frac{A_i}{||\mathbf{v}||}\left(\mu_i\mathbf{n} - \mathbf{b}\right) \text{ and } \lambda = -\frac{A_i\mu_i}{||\mathbf{v}||}\mathcal{H}_{bt} \tag{82}$$

Eq.(81) amounts to

$$\partial_b\alpha - \mu_i\partial_n\alpha = \mu_i\mathcal{H}_{bt} \quad i = 1, 2. \tag{83}$$

Since $\mu_2 - \mu_1 \neq 0$ we simply get

$$\partial_n\alpha = -\mathcal{H}_{bt}, \qquad \partial_b\alpha = 0. \tag{84}$$

To find a solution to (84), note that, for any function $\varphi(s)$, the derivative $\partial_s\varphi(s)$ satisfies

$$\begin{aligned} \partial_n\partial_s\varphi(s) &= \mathcal{H}_{bt}\partial_s\varphi(s) && (85)\\ \partial_b\partial_s\varphi(s) &= -\mathcal{H}_{nt}\partial_s\varphi(s). && (86) \end{aligned}$$

According to Proposition 1 above, $\mathcal{H}_{nt} = 0$ for the cases 1 and 2bii of Frenet-Serret frames that can be applied to gradient systems. Therefore, we have

$$\begin{aligned} \partial_n\partial_s\varphi(s) &= \mathcal{H}_{bt}\partial_s\varphi(s) && (87)\\ \partial_b\partial_s\varphi(s) &= 0 \end{aligned}$$

and choosing

$$\alpha = \frac{1}{\partial_s\varphi(s)} \tag{88}$$

solves Eq.(81). It is possible to get a more specific solution using Eqs.(32) and (33) which state that

$$||\mathbf{v}|| = \partial_s F(s). \tag{89}$$

Therefore one may choose

$$\alpha = \frac{1}{||\mathbf{v}||} \tag{90}$$

which proves the proposition ■

Above proposition proves that the arclengths of integral curves of vector fields

$$\frac{1}{||\mathbf{v}||}\mathbf{t}, \ \frac{1}{||\mathbf{v}||}\mathbf{J}_i \times \mathbf{t}$$

provide parametrizations and hence a local coordinate system for the potential surface $H_i = c$.

5 Examples

5.1 Sphere

Let $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. Consider the gradient system

$$\frac{d\mathbf{r}\,(t)}{dt} = \mathbf{r}\,(t) \tag{91}$$

with the potential function

$$F\,(\mathbf{r}) = \frac{r^2}{2} \tag{92}$$

where level surfaces are spheres. To construct the Frenet-Serret frame, we begin with the unit tangent vector

$$\mathbf{t} = \frac{\mathbf{r}}{r} \tag{93}$$

and take normal and binormal vectors as the lines of curvature of level surfaces of the potential function. Since level surfaces are spheres, which are surfaces of revolution, their lines of curvatures are latitudes and longitudes. Adapting spherical coordinates

$$\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned} \tag{94}$$

the Frenet-Serret frame can be written as

$$\begin{aligned} \mathbf{t} &= \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \\ \mathbf{n} &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \\ \mathbf{b} &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k}. \end{aligned} \tag{95}$$

For the construction of Hamiltonian functions, we will fix the unit sphere. The distance function on the unit sphere is

$$d\,(P_i, P) = \arccos\left(\mathbf{P}_i{\cdot}\mathbf{P}\right) \tag{96}$$

where $\mathbf{P}$ and $\mathbf{P}_i$ are position vectors of the points P and P_i. Written explicitly, if

$$\mathbf{P}_i = (\sin\phi_i \cos\theta_i, \sin\phi_i \sin\theta_i, \cos\phi_i) \tag{97}$$

$$\mathbf{Q} = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi) \tag{98}$$

then

$$d(P, P_i) = \arccos\left(\cos\phi \cos\phi_i + \sin\phi \sin\phi_i \cos(\theta - \theta_i)\right). \tag{99}$$

Now choosing the north pole $P_1 = (0, 0, 1)$, i.e. $\phi_1 = 0$ and $\theta_1 = 0$, as our fixed point, we get

$$d(P_1, P) = \phi \tag{100}$$

and choosing the point on the equator $P_2 = (1, 0, 0)$, i.e. $\phi_2 = \pi/2$ and $\theta_2 = 0$, the distance becomes

$$d(P_2, P) = \arccos(\sin\phi \cos\theta). \tag{101}$$

To extend these two functions to $\mathbb{R}^3$, we define the Hamiltonian functions

$$H_1(\mathbf{r}(t)) = H_1\left(\frac{\mathbf{r}(t)}{r(t)}\right) = d\left(P_1, \frac{\mathbf{r}(t)}{r(t)}\right) = \phi \tag{102}$$

$$H_2(\mathbf{r}(t)) = H_2\left(\frac{\mathbf{r}(t)}{r(t)}\right) = d\left(P_2, \frac{\mathbf{r}(t)}{r(t)}\right) = \arccos(\sin\phi \cos\theta). \tag{103}$$

These functions can be written in terms of y/x and z/x as

$$H_1(\mathbf{r}(t)) = \arccos\left(\frac{1}{\sqrt{(x/z)^2 + (y/x)^2 (x/z)^2 + 1}}\right) \tag{104}$$

$$H_2(\mathbf{r}(t)) = \arccos\left(\frac{x}{\sqrt{1 + (y/x)^2 + (z/x)^2}}\right). \tag{105}$$

Indeed, the functions

$$h_1(x, y, z) = \frac{y}{x}, \; h_2(x, y, z) = \frac{z}{x}$$

are the fundamental conserved quantities of Eq.(91).

5.2 A Linear Poisson system

Consider the gradient system defined by the vector field

$$\mathbf{v}(\mathbf{x}) = (yz, xz, xy) \tag{106}$$

which is an unphysical version of the Euler top. The potential surfaces are

$$F(x, y, z) = xyz = c \tag{107}$$

As mentioned in [38], the geodesic flows on potential surfaces for $c \neq 0$ are not integrable, and hence one may not expect to find simple Hamiltonian functions for this system. However, letting $c = 0$ yields

$$xyz = 0 \tag{108}$$

which is nothing but the non-smooth union of coordinate planes. On this surface consider the closed subset

$$x \geq 0, \ \ y \geq 0, \ \ z = 0 \tag{109}$$

which is the first quadrant of $xy-$ plane. Then, the Frenet-Serret frame becomes

$$(\mathbf{t}, \mathbf{n}, \mathbf{b}) = (\mathbf{k}, \mathbf{i}, \mathbf{j}) \tag{110}$$

Choosing $P_1 = (0, 0)$ and $P_2 = (1, 0)$ on the $xy-$plane it is possible to define two distance functions

$$d\left(P_1, P\right) = \sqrt{x^2 + y^2} \tag{111}$$

$$d\left(P_2, P\right) = \sqrt{\left(x-1\right)^2 + y^2} \tag{112}$$

whose gradients are

$$\nabla d\left(P_1, P\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) \tag{113}$$

$$\nabla d\left(P_2, P\right) = \left(\frac{x-1}{\sqrt{\left(x-1\right)^2+y^2}}, \frac{y}{\sqrt{\left(x-1\right)^2+y^2}}\right) \tag{114}$$

Using the immediate condition

$$\frac{d}{dt}\left(x\left(t\right)y\left(t\right)z\left(t\right)\right) = 0 \tag{115}$$

we obtain the time evolution of z coordinate

$$\frac{dz\left(t\right)}{dt} = -\frac{z\left(t\right)}{x\left(t\right)}\frac{dx\left(t\right)}{dt} - \frac{z\left(t\right)}{y\left(t\right)}\frac{dy\left(t\right)}{dt}. \tag{116}$$

That means, we can extend to the gradient vector fields in Eqs.(113) and (115) to the vector fields

$$\mathbf{u}_1\left(\mathbf{x}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x, y, -2z\right) \tag{117}$$

$$\mathbf{u}_2\left(\mathbf{x}\right) = \frac{1}{\sqrt{\left(x-1\right)^2+y^2}}\left(x-1, y, -2z+\frac{z}{x}\right) \tag{118}$$

which form a frame for the tangent space of the potential surface $xyz = 0$. Using the decomposition

$$\mathbf{u}_2\left(\mathbf{x}\right) = \frac{1}{\sqrt{\left(x-1\right)^2 + y^2}}\left(x, y, -2z\right) + \frac{1}{x\sqrt{\left(x-1\right)^2 + y^2}}\left(-x, 0, z\right) \tag{119}$$

it is possible find another frame consisting of gradient vector fields

$$\nabla H_1 = (x, y, -2z) \tag{120}$$

$$\nabla H_2 = (-x, 0, z) \tag{121}$$

for the functions

$$H_1\left(x, y, z\right) = \frac{1}{2}\left(x^2 + y^2 - 2z^2\right) \tag{122}$$

$$H_2\left(x, y, z\right) = \frac{1}{2}\left(z^2 - x^2\right) \tag{123}$$

which are Hamiltonian functions of the gradient system defined by Eq.(106).

5.3 The Aristotelian Model of Three Body Motion

In [39], we showed that the Aristotelian model of three-body motion is defined by the gradient vector field

$$\mathbf{v}(\mathbf{x}) = \left(\frac{c}{x-y} + \frac{b}{x-z}, \frac{a}{y-z} + \frac{c}{y-x}, \frac{b}{z-x} + \frac{a}{z-y}\right) \tag{124}$$

where the potential surfaces are

$$F\left(\mathbf{x}\right) = a\ln\left(y-z\right) + b\ln\left(x-z\right) + c\ln\left(x-y\right) = K. \tag{125}$$

We will choose the level surface defined by the constant

$$K = a + b + c = 1 \tag{126}$$

and parametrized by

$$u = \frac{x-z}{y-z} - \frac{1}{2}, \quad v = y - z \tag{127}$$

so that the potential surface in new coordinates will be

$$\ln v\left(u + \frac{1}{2}\right)^b\left(u - \frac{1}{2}\right)^c = 1. \tag{128}$$

We can solve the variable v

$$v = f\left(u\right) = \frac{e}{\left(u + \frac{1}{2}\right)^b\left(u - \frac{1}{2}\right)^c} \tag{129}$$

and obtain the relation

$$\frac{f'(u)}{f(u)} = -\left(\frac{b}{u+\frac{1}{2}} + \frac{c}{u-\frac{1}{2}}\right).$$

With the parameters (u, z), the potential surface becomes

$$\mathbf{X}(u,z) = f(u)\left(u+\frac{1}{2}, 1, 0\right) + z(1,1,1) \tag{130}$$

which is a ruled surface. To obtain the orthogonal parametrization, let

$$\mathbf{e}_1 = \frac{1}{\sqrt{3}}(1,1,1), \ \mathbf{e}_2 = \frac{1}{\sqrt{6}}(2,-1,-1), \ \mathbf{e}_3 = \frac{1}{\sqrt{2}}(0,1,-1) \tag{131}$$

and define

$$\boldsymbol{\alpha}(u) = \frac{\sqrt{2}uf(u)}{\sqrt{3}}\mathbf{e}_2 + \frac{f(u)}{\sqrt{2}}\mathbf{e}_3 \tag{132}$$

$$w = \sqrt{3}z + \frac{1}{\sqrt{3}}f(u)\left(u+\frac{3}{2}\right) \tag{133}$$

Then, we have

$$\mathbf{X}(u,w) = \boldsymbol{\alpha}(u) + w\mathbf{e}_1. \tag{134}$$

The fundamental forms are

$$(g_{ij}) = \begin{pmatrix} \|\boldsymbol{\alpha}'(u)\|^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{135}$$

$$(l_{ij}) = \frac{1}{\|\nabla F\|}\begin{pmatrix} \boldsymbol{\alpha}''(u)\cdot\nabla F & 0 \\ 0 & 0 \end{pmatrix} \tag{136}$$

where

$$\nabla F = -\frac{\sqrt{3}f'(u)}{\sqrt{2}f^2(u)}\mathbf{e}_2 + \frac{\sqrt{2}\left(f(u)+uf'(u)\right)}{f^2(u)}\mathbf{e}_3 \tag{137}$$

$$\boldsymbol{\alpha}'(u) = \frac{\sqrt{2}\left(uf'(u)+f(u)\right)}{\sqrt{3}}\mathbf{e}_2 + \frac{f'(u)}{\sqrt{2}}\mathbf{e}_3. \tag{138}$$

Note also that

$$\nabla F = \frac{\sqrt{3}}{f^2(u)}\mathbf{e}_1 \times \boldsymbol{\alpha}'(u).$$

Since the Gaussian curvature of this ruled surface is zero, the potential surfaces are developable surfaces, and therefore can be mapped isometrically onto the plane. We may choose the Frenet-Serret frame on this surface to be the Darboux frame

$$(\mathbf{t},\mathbf{n},\mathbf{b}) = \left(\frac{\nabla F}{\|\nabla F\|}, \mathbf{e}_1, \frac{\boldsymbol{\alpha}'(u)}{\|\boldsymbol{\alpha}'(u)\|}\right) \tag{139}$$

of lines of curvature which are the coordinate curves $u = c_1$ and $w = c_2$. Since ∇F, $\boldsymbol{\alpha}'(u)$ and $\boldsymbol{\alpha}''(u)$ are vectors in the plane spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$, we have

$$\nabla F \cdot (\boldsymbol{\alpha}'(u) \times \boldsymbol{\alpha}''(u)) = 0$$

and since $\mathbf{e}_1$ is a constant vector, it is easy to see that these lines of curvature are also geodesics. Therefore, the coordinates (s, n, b) are geodesic distances in $\mathbf{t}, \mathbf{n}, \mathbf{b}$ directions, respectively. The metric in the new coordinate system becomes Euclidean. This allows us to write

$$(\mathbf{n}, \mathbf{b}) = (\nabla H_1(n), \nabla H_2(b)) \tag{140}$$

where H_1 and H_2 are geodesic distance functions in directions of $\mathbf{n}$ and $\mathbf{b}$. In coordinates (u, v, w), the first Hamiltonian function is

$$H_1(w) = \int_{w_0}^{w} \|\mathbf{e}_1\| \, dt = w \tag{141}$$

and it is easy to see that

$$w = \frac{1}{\sqrt{3}}(x + y + z). \tag{142}$$

For the second Hamiltonian function, we have

$$H_2(u) = \int_{u_0}^{v} \|\boldsymbol{\alpha}'(t)\| \, dt = \int_{u_0}^{u} \sqrt{\frac{2}{3}(tf'(t) + f(t))^2 + \frac{1}{2}(f'(t))^2} dt \tag{143}$$

or equivalently

$$H_2(x, y, z) = \sqrt{\frac{2}{3}} \int_{u_0}^{\frac{2x-y-z}{2(y-z)}} \sqrt{\left(t + \frac{f(t)}{f'(t)}\right)^2 + \frac{3}{4}} df(t).$$

6 Conclusion

We developed differential calculus in Frenet-Serret frame. We extended the result of [27] for constructing Frenet-Serret frame to all dynamical systems with the help of eigenvectors of curl operator and a result of Chandrasekhar and Kendall in [28]. Considering bi-Hamiltonian structure and Jacobi identity in Frenet-Serret frame associated to a dynamical system, we proved that all dynamical systems in three dimensions possess two compatible Poisson structures. We also presented the relation between Hamiltonian functions and Poisson vectors.

Given a gradient dynamical system, we presented the geometric parameters of both level surface and an arbitrary curve on it. In particular, we considered, on level surfaces of potential function, gradient flows of restrictions of Hamiltonian functions and proved that it is possible to find Hamiltonian functions

whose gradient flows on level surface have geodesic curvature zero. This result led us to show that Hamiltonian functions are determined by distance functions, namely, geodesic lenghts from an arbitrary point to two different fixed points on the level surface of potential function.

Finally, by means of transformations bringing one of the components of an orthogonal metric to constant, we proved that finding two Hamiltonian functions of a gradient system is the same as constructing geodesic coordinates of its potential surfaces. As examples, we worked out decoupled flow of radius vector of a sphere, a quadratic dynamical system possessing linear Poisson structures, and the Aristotelian model of three body motion.

References

[1] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Diff. Geom. 12 (1977) 253-300.

[2] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523-557.

[3] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer-Verlag, New York, 1993.

[4] J. E. Marsden, T. Ratiu, Introduction to Symmetry and Mechanics, Texts in Applied Mathematics, Vol.17, (Springer, Berlin, 1994).

[5] P. J. Morrison, Hamiltonian description of the ideal fluid, Rev. Mod. Phys. 70 (1998) 467-521.

[6] Y. Nambu, Generalized Hamiltonian Dynamics, Phys. Rev. D 7 (1973) 2405-2412.

[7] F. Bayen, M. Flato, Remarks concerning Nambu‘s generalized mechanics, Phys. Rev. D 11 (1975) 3049-3053.

[8] N. Mukunda, E. Sudarshan, Relation between Nambu and Hamiltonian mechanics, Phys. Rev. D 13 (1976) 2846-2850.

[9] H. Hirayama, Realization of Nambu mechanics: a particle interacting with an SU(2) monopole, Phys. Rev. D 16 (1977) 530-532.

[10] M. Kus, Integrals of motion for the Lorenz system, J. Phys. A: Math. Gen. 16 (1983) L689-L691.

[11] J. M. Strelcyn and S. Wojciechowski, A method of finding integrals for three-dimensional dynamical systems, Phys. Lett. A133 (1988) 207-212.

[12] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J. M. Strelcyn and S. Wojciechowski, Integrals of quadratic ordinary differential equations in $\mathbb{R}^3$: The Lotka-Volterra system, Physica A 163 (1990) 683-722.

[13] Y. Nutku, Hamiltonian structure of the Lotka-Volterra equations, Phys. Lett. A, 145 (1990) 27-28.

[14] Y. Nutku, Bi-Hamiltonian structure of the Kermack-McKendrick model for epidemics, J. Phys. A: Math. Gen. 23 (1990) L1145-L1146.

[15] H. J. Giacomini, C. E. Repetto and O. P. Zandron, Integrals of motion for three-dimensional non-Hamiltonian dynamical systems, J. Phys. A: Math. Gen. 24 (1991) 4567-4574.

[16] S. A. Hojman, Quantum algebras in classical mechanics, J. Phys. A: Math. Gen., 24 (1991) L249-L254.

[17] H. Gümral and Y. Nutku, Poisson structure of dynamical systems with three degrees of freedom, J.Math.Phys. 34 (1993) 5691-5723.

[18] F. Haas and J. Goedert, On the Generalized Hamiltonian Structure of 3D Dynamical Systems, Phys. Lett. A 199 (1995) 173-179.

[19] G. R. W. Quispel and H. W. Capel, Solving ODEs numerically while preserving a first integral, Phys. Lett. A218 (1996) 223-228.

[20] B. Hernández-Bermejo, A constant of motion in 3D implies a local generalized Hamiltonian structure, Phys. Lett. A234 (1997) 35-40.

[21] A. Ay, M. Gürses and K. Zheltukhin, Hamiltonian Equations in $\mathbb{R}^3$, J. Math. Phys. 44 (2003) 5688-5705.

[22] B. Hernández-Bermejo and V. Fairén, Separation of variables in the Jacobi identities, Phys. Lett. A 271 (2000) 258–263.

[23] B. Hernández-Bermejo, New solutions of the Jacobi equations for three-dimensional Poisson structures, J. Math. Phys. 42 (2001) 4984-4996.

[24] B. Karasözen, Poisson integrators, Mathematical and Computer Modelling, 40 (2004) 1225-1244.

[25] B. Hernández-Bermejo, Characterization and global analysis of a family of Poisson structures, Phys. Lett. A 355 (2006) 98–103.

[26] B. Hernández-Bermejo, New solution family of the Jacobi equations: Characterization, invariants, and global Darboux analysis, J. Math. Phys. 48 (2007) 022903-022914.

[27] E. Abadoğlu and H. Gümral, Bi-Hamiltonian Structure in Frenet-Serret Frame, Physica D 238 (2009) 526-530.

[28] S. Chandrasekhar and P. C. Kendall, On Force-Free Magnetic Fields, Astrophysical Journal 126 (1957) 457-460.

[29] V. I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, Springer-Verlag, New York, 1983.

[30] P. Tondeur, Foliations on Riemannian Manifolds, Springer, Berlin, 1988.

[31] B. L. Reinhart, Differential Geometry of Foliations, Springer, Berlin, 1983.

[32] F. Magri, A simple model of the integrable Hamiltonian Equation. J. Math. Phys. 19, (1978) 1156-1162.

[33] H. E. Moses, Eigenfunctions of the Curl Operator, Rotationally Invariant Helmholtz Theorem, and Applications to Electromagnetic Theory and Fluid Mechanics, SIAM J. Appl. Math. 21 (1971) 114-144.

[34] I. M. Benn and J. Kress, Force-free fields from Hertz potentials, J. Phys. A: Math. Gen. 29 (1996) 6295-6304.

[35] L.C. Garcia de Andrade, Topology of magnetic helicity in torsioned filaments in Hall plasmas, astro-ph/07104594

[36] F. Calogero, D. Gomez-Ullate, P. M. Santini, M. Sommacal, The transition from regular to irregular motions, explained as travel on Riemann surfaces, J. Phys. A: Math. Gen. 38 (2005) 8873-8896.

[37] H. Hopf, W. Rinow, "Ueber den Begriff der vollständigen differentialgeometrischen Fläche". Commentarii Mathematici Helvetici 3 (1931), 209–225. doi:10.1007/BF01601813

[38] T. J. Waters, Non-integrability of geodesic flow on certain algebraic surfaces, Phys. Lett. A 376 (2012) 1442–1445. doi: 10.1016/j.physleta.2012.03.016

[39] E. Abadoğlu , H. Gümral, Poisson structures for the Aristotelian model of three-body motion, J. Phys. A: Math. Theor. 44 (2011) 325204. doi:10.1088/1751-8113/44/32/325204